THE PRIMITIVE AND DIAMOND SURFACES LOCALLY MINIMIZE THE VARIANCE OF GAUSSIAN CURVATURE

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ABSTRACT. In response to a conjecture from material sciences, we prove that Schwarz's Primitive (P) and Diamond (D) surfaces are local minimizers of the variance of Gaussian curvature among local deformations within the moduli space of embedded triply periodic minimal surfaces of genus 3 (TPMSg3s). Our approach interprets the branch values of the Gauss map as a configuration of eight points on the sphere and expresses the variance of Gaussian curvature as the product of two integrals involving exponentials of Green's functions. We then show that the Hessian of this functional is positive definite at the cubic configuration when restricted to antipodal deformations, thereby establishing the local minimality of the P and D surfaces. Along the other deformation directions, the Hessian is positive definite except for a two-dimensional eigenspace with slightly negative eigenvalues, leaving promising hope that the Gyroid is also a local minimizer.

Triply Periodic Minimal Surfaces (TPMSs) are minimal surfaces in flat 3-tori. The genus of a TPMS is at least three. A TPMS of genus three (TPMSg3) has a Weierstrass parameterization defined on a branched double cover of the sphere with eight branch points. The branched covering map is provided by the Gauss map, so the branch points correspond to eight *flat points* on the TPMS where the Gaussian curvature vanishes. We refer the reader to [5] for general reference.

TPMSs find applications in natural sciences as a model for periodic bicontinuous structures [3]. In nature and in laboratories, the chance to observe Schwarz's Primitive (P), Diamond (D) and Schoen's Gyroid (G) surfaces is much higher than finding other TPMSs. Note that these surfaces belong to the same Bonnet family. They are therefore locally isometric, have the same branch points, and the same variance of Gaussian curvature. For all the three surfaces, the branch points are vertices of a cube. Based on numerical evidence [6], it was conjectured that

Conjecture. *Schwarz's Primitive, Diamond, and Schoen's Gyroid surfaces have the minimal variance of Gaussian curvature among all embedded Triply Periodic Minimal Surfaces of genus 3.*

In this short note, we prove that the Primitive and Diamond are indeed local minimizers.

Theorem. *Schwarz's Primitive and Diamond surfaces are local minimizers for the variance of Gaussian curvature among local deformations within the moduli space of embedded Triply Periodic Minimal Surfaces of genus 3.*

Our approach extends the variance of Gaussian curvature to an energy defined on the 13-dimensional space of configurations of eight points on the sphere up to global rotations. Only a five-dimensional locus of these configurations corresponds to embedded TPMSg3s. More specifically, for an embedded TPMSg3, the eight

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points correspond to the normal vectors at the eight flat points. In particular, the five-dimensional subspace of antipodal configurations corresponds to embedded TPMSg3s known as the *Meeks family*. For Diamond, Primitive, and Gyroid surfaces, the eight points are at the vertices of a cube.

In [4], it was proved that, up to homotheties and Euclidean isometries, the Primitive, Diamond, and Gyroid surfaces all belong to a unique locally rigid 5-parameter smooth family of pairwise non-homothetic embedded TPMSg3s. In particular, the family that contains the Diamond and the Primitive surfaces is the Meeks family. Schoen's Gyroid, however, does not belong to Meeks family.

At the cubic configuration, we compute the Hessian of our energy with respect to local deformations of the configuration. We find, unfortunately, that the Hessian has a 2-dimensional negative eigenspace generated by "twists". So the Hessian is indefinite, and the cubic configuration is not a local minimizer among point configurations. Nevertheless, we are able to prove positive definiteness on the subspace of antipodal deformations. So the cubic configuration is a local minimizer among antipodal configurations, thereby proving the Theorem.

We are unable to conclude on the Gyroid because its 5-parameter family of local deformations is not fully understood. Among the known deformations of the Gyroid, it is indeed a local minimizer. But in the presence of the negative eigenspace, we cannot rule out the possibility that it might not be a local minimizer along other deformations.

Remark 1. Our energy generalizes to the configurations of n points on the sphere. See Section 1.5 for details. This is an interesting problem for discrete geometry in its own right.

The paper is organized as follows. In Section 1, we will reformulate the problem in terms of point configurations on the sphere. The Theorem is proved in Section 2, but many complicated computational details are postponed to the appendix. We compute the gradient of our energy in Appendix A, the Hessian in Appendix B, and analyze the definiteness of the Hessian on each of irreducible representations in Appendix C.

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1. Reformulation

1.1. **Weierstrass parameterization.** Given a TPMSg3, let $p_1, \cdots, p_8 \in \mathbb{C}$ be (the stereographic projection of) the Gauss map at the eight flat points. Consider the hyperelliptic Riemann surface Σ of genus three defined by

$$w^2 = P(z) = \prod_{i=1}^{8}(z - p_i).$$

It can be seen as a branched double cover of the Riemann sphere $\mathbb{S}^2 \simeq \mathbf{CP}^1$ with eight branch points. Then we have the following *Weierstrass parameterization* for the TPMS (up to translations)

$$\Sigma \ni (z, w) \mapsto \operatorname{Re} \int^{(z,w)} \frac{(1 - z^2, \mathrm{i}(1 + z^2), 2z)}{w} e^{i\vartheta}\, \mathrm{d}z. \tag{1}$$

Switching the two branches by $(z, w) \mapsto (z, -w)$ corresponds to the inversion with respect to any of the eight flat points. The parameter ϑ is the *Bonnet angle*; changing ϑ gives a locally isometric deformation of the surface known as *Bonnet rotation*. Two surfaces whose ϑ differ by $\pi/2$ are called *conjugate pairs*.

For the minimal surface to be triply periodic and globally well-defined, one needs to solve the period problem (aka, *close the periods*). More precisely, one finds p_i's and ϑ so that the real parts of the integrals over closed cycles on Σ form a three dimensional lattice. The period problem involves three real equations for each of the three coordinates, hence nine real equations in total. Since we have 17 real parameters (p_i's and ϑ), the set of TPMSg3s forms a 8-dimensional variety up to translations and homotheties, or 5-dimensional variety up to Euclidean isometries and homotheties.

Closing the periods is usually very difficult. But Meeks proved in [5] that, when the branch points form four antipodal pairs, there exists a conjugate pair of embedded TPMSg3s. Up to homotheties and Euclidean isometries, these TPMSg3s form a 5-dimensional manifold termed *Meeks family*.

In this paper, we will not insist on closing the periods. The variance of Gaussian curvature is invariant under local isometries such as Bonnet rotations. So it is a function defined on the 13-dimensional space of configurations of eight points on the sphere up to rotations. The Meeks surfaces correspond to the five-dimensional locus of antipodal configurations, for which we know the existence of ϑ that closes the periods. Our main Theorem is equivalent to that the cubic configuration is a local minimizer of the Gauss variance on this 5-dimensional subspace.

1.2. **Variance of Gaussian curvature.** More precisely, we want to minimize the (dimensionless) variance of Gauss curvature, defined by

$$\sigma(K) = \frac{\langle K^2 \rangle - \langle K \rangle^2}{\langle K \rangle^2} = \left[\frac{\int_\Sigma K^2 \,\mathrm{d}A}{\int_\Sigma \mathrm{d}A} - \left(\frac{\int_\Sigma K \,\mathrm{d}A}{\int_\Sigma \mathrm{d}A} \right)^2 \right] \Big/ \left(\frac{\int_\Sigma K \,\mathrm{d}A}{\int_\Sigma \mathrm{d}A} \right)^2.$$

By Gauss-Bonnet, $\int_\Sigma K \,\mathrm{d}A$ is a topological invariant. It is then equivalent to minimize

$$\int_\Sigma K^2 \mathrm{d}A \times \int_\Sigma \mathrm{d}A. \tag{2}$$

From the Weierstrass parameterization, we compute

$$\mathrm{d}A = \frac{(1+|z|^2)^2}{|w|^2} \, dx \wedge dy, \qquad K = -4 \frac{|w|^2}{(1+|z|^2)^4},$$

where $z = x + y\mathrm{i}$. Indeed, one verifies that

$$\int_\Sigma K \mathrm{d}A = -4 \int_\Sigma \frac{dx \wedge dy}{(1+|z|^2)^2} = -8\pi$$

as expected. The area element and the Gaussian curvature are invariant after switching the branches. It then suffices to compute the invariance on one of the branches. So (2) becomes (up to a constant factor)

$$\int_{\mathbf{CP}^1} \frac{|P(z)|}{(1+|z|^2)^6} \, dx \wedge dy \times \int_{\mathbf{CP}^1} \frac{(1+|z|^2)^2}{|P(z)|} \, dx \wedge dy. \tag{3}$$

Note that the second integral is improper: The integrand is not defined at the zeros of $P(z)$. So the integral is defined as the limit

$$\int_{\mathbf{CP}^1} \frac{(1+|z|^2)^2}{|P(z)|} \, dx \wedge dy = \lim_{\varepsilon \to 0} \int_{\mathbf{CP}^1 \setminus \cup_i D^i_\varepsilon} \frac{(1+|z|^2)^2}{|P(z)|} \, dx \wedge dy,$$

where D_ε^i denote the ε-disks around p_i under the Fubini–Study metric. Fortunately, the limit converges.

1.3. **Fubini–Study metric.** Note that

$$\mathrm{d}A_{\mathrm{FS}} = \frac{4\,dx \wedge dy}{(1+|z|^2)^2}$$

is the area element for the Fubini–Study metric on $\mathbf{CP}^1$, which is (normalized to) exactly the standard metric on the unit sphere $\mathbb{S}^2$. Moreover, we can identify $P(z)$ to a global section s of the holomorphic line bundle $\mathcal{O}(8) = \mathcal{O}(1)^{\otimes 8}$ over $\mathbf{CP}^1$, and

$$\|s\|_{\mathrm{FS}} = \frac{|P(z)|}{(1+|z|^2)^4}$$

is its point-wise Fubini–Study norm. See, for instance, [2]. Then (3) becomes (up to a constant factor)

$$J_8 := \int_{\mathbf{CP}^1} \|s\|_{\mathrm{FS}}\,\mathrm{d}A_{\mathrm{FS}} \times \int_{\mathbf{CP}^1} \|s\|_{\mathrm{FS}}^{-1}\,\mathrm{d}A_{\mathrm{FS}}. \tag{4}$$

Note again that the second integral is improper. It is defined as the convergent limit of the integral over $\mathbf{CP}^1 \setminus \cup_i D_\varepsilon^i$ as $\varepsilon \to 0$.

1.4. **Lelong–Poincaré Equation.** Define

$$u = \log \|s\|_{\mathrm{FS}}.$$

Note that the Fubini–Study metric on $\mathcal{O}(8)$ is Hermitian. One then easily verifies the Lelong–Poincaré Equation

$$\Delta u = 2\pi \sum_{i=1}^{8} \delta(z - p_i) - 4\frac{4}{(1+|z|^2)^2},$$

or, using the Laplacian under the Fubini–Study metric,

$$\Delta_{\mathrm{FS}} u = \frac{(1+|z|^2)^2}{4}\Delta u = 2\pi \sum_{i=1}^{8} \delta_i(z) - 4,$$

where

$$\delta_i(z) = \frac{(1+|z|^2)^2}{4}\delta(z - p_i)$$

denote the Dirac delta function under the Fubini–Study metric centered at p_i.

The solution is given by

$$u = 2\pi \sum_{i=1}^{8} G_i(z),$$

where $G_i(z)$ is the Green function solving

$$\Delta_{\mathrm{FS}} G_i(z) = \delta_i(z) - \frac{1}{4\pi}. \tag{5}$$

On the unit sphere $\mathbb{S}^2 \simeq \mathbf{CP}^1$, we have

$$G_i(z) = \frac{1}{2\pi} \log \sin \frac{\mathrm{dist}(z, p_i)}{2} + C \tag{6}$$

where $\mathrm{dist}(z, p_i)$ is the spherical distance between z and p_i. Then (4) becomes

$$J_8 = \int_{\mathbb{S}^2} \exp(u)\,\mathrm{d}A \times \int_{\mathbb{S}^2} \exp(-u)\,\mathrm{d}A, \tag{7}$$

where $\mathrm{d}A$ is the standard spherical area element on the unit sphere. Note that (7) does not depend on the constant C in G_i, so we assume $C = 0$ from now on. Note again that the second integral is improper and is defined as the convergent limit

$$\int_{\mathbb{S}^2} \exp(-u)\,\mathrm{d}A = \lim_{\varepsilon\to 0} \int_{\Omega_\varepsilon} \exp(-u)\,\mathrm{d}A,$$

where

$$\Omega_\varepsilon = \mathbb{S}^2 \setminus \cup_{i=1}^n D_\varepsilon^i.$$

The Conjecture would be proved if J_8 is minimized among all configurations when $p_i \in \mathbb{S}^2$ are the vertices of a cube. This is, unfortunately, not the case, as we will see later.

Remark 2. With this reformulation, we actually extend the variance of Gaussian curvature beyond embedded TPMSg3s. We are now considering all minimal surfaces with the Weierstrass parameterization (1). In particular, we do not need the period problems to be solved, not to mention the embeddedness. The variance can be seen as an energy function defined on the space of point configurations.

1.5. **Generalization.** The analysis above can be generalized as follows. Fix n distinct points $p_1, \cdots, p_n$, and define the polynomial

$$P(z) = \prod_{i=1}^n (z - p_i).$$

Consider the product of integrals

$$\int_{\mathbf{CP}^1} \frac{|P(z)|}{(1+|z|^2)^{\frac{n}{2}+2}}\,dx\wedge dy \times \int_{\mathbf{CP}^1} \frac{(1+|z|^2)^{\frac{n}{2}-2}}{|P(z)|}\,dx\wedge dy.$$

Note that

$$\|s\|_{\mathrm{FS}} = \frac{|P(z)|}{(1+|z|^2)^{n/2}}$$

is the point-wise Fubini–Study norm of $P(z)$ as a global section s of $\mathcal{O}(n)$ over $\mathbf{CP}^1$. Then the product can be rewritten (up to a constant factor) into the form

$$J_n := \int_{\mathbf{CP}^1} \|s\|_{\mathrm{FS}}\,\mathrm{d}A_{\mathrm{FS}} \times \int_{\mathbf{CP}^1} \|s\|_{\mathrm{FS}}^{-1}\,\mathrm{d}A_{\mathrm{FS}}.$$

Now the Lelong–Poincaré Equation is

$$\Delta_{\mathrm{FS}} u = 2\pi \sum_{i=1}^n \delta_i(z) - n/2,$$

which is solved by

$$u = 2\pi \sum_{i=1}^n G_i(z).$$

So

$$J_n = \int_{\mathbb{S}^2} \exp(u)\,\mathrm{d}A \times \int_{\mathbb{S}^2} \exp(-u)\,\mathrm{d}A.$$

And we could ask the following problem

Problem. *What are the positions of* $p_1, \cdots, p_n$ *that minimize* J_n*?*

This is an interesting problem in its own right. When $n = 8$, this is exactly the problem of finding branch values to minimize the variance of the Gauss curvature.

Remark 3. J_n can be interpreted as a nonlinear measure of the variance, or contrast, of the potential obtained by superimposing n Green potentials, treating peaks and valleys symmetrically.

2. Proof

We now present our proof to the Theorem. Computational details are left to the appendix.

Proof. Recall that

$$\Omega_\varepsilon = \mathbb{S}^2 \setminus \cup_{i=1}^n D_\varepsilon^i,$$

where D_ε^i denote the ε-disks around p_i under the spherical metric. Define

$$I_\varepsilon^\pm = \int_{\Omega_\varepsilon} \exp(\pm u)\, \mathrm{d}A,$$

and two probability measure $\mathrm{d}\mu_\varepsilon^\pm = \rho_\varepsilon^\pm \mathrm{d}A$ on Ω_ε, where

$$\rho_\varepsilon^\pm = \frac{\exp(\pm u)}{\int_{\Omega_\varepsilon} \exp(\pm u)\, \mathrm{d}A} = \frac{\exp(\pm u)}{I_\varepsilon^\pm}.$$

We need to compute the gradient and the Hessian of

$$\log J_8 = \log I^+ + \log I_-.$$

In Appendix A we compute the gradient of $\log I^\pm$ as follows:

$$\begin{aligned} \langle \nabla_p \log I^\pm, v\rangle &:= \lim_{\varepsilon\to 0} \langle \nabla_p \log I_\varepsilon^\pm, v\rangle \\ &= \int_{\mathbb{S}^2} \sum_{i=1}^n \langle \mp 2\pi \nabla_z G_i, \tilde{v}_i\rangle \mathrm{d}\mu^\pm, \end{aligned} \tag{8}$$

where

$$v = (v_1, \cdots, v_8) \in \bigoplus_{i=1}^8 T_{p_i}\mathbb{S}^2,$$

and $\tilde{v}_i$ is a Killing field such that $\tilde{v}_i(p_i) = v_i$. Note that the formula is independent of the choice of the Killing field. To see this, assume another choice $\tilde{v}_i'$. Then $\tilde{w}_i = \tilde{v}_i' - \tilde{v}_i$ is a Killing field that vanishes at p_i, which is given as a rotation around p_i. Consequently, we have $\langle 2\pi\nabla_z G_i, \tilde{w}_i\rangle = 0$. Moreover, if $\tilde{v}_i = \tilde{v}_*$ is the same Killing field for all i, one easily verifies that the gradient vanishes. This is expected because J is invariant under global rotations.

At the cubic configuration, the gradient may be identified with a tuple of eight tangent vectors, one at each p_i. The O_h-invariance of $I^\pm$ implies that this tuple is invariant under the induced action of O_h, including its permutation of the vertices. The tangent representation has no nonzero invariant vector, so the gradient must vanish at the cube. Therefore, the cube configuration is a critical point of $\log I^\pm$ and of $\log J_8$.

Then in Appendix B, we compute the Hessian of $\log I^\pm$ at a critical point as follows:

$$\begin{aligned} \nabla_p^2 \log I^\pm(v, v) &:= \lim_{\varepsilon\to 0} \nabla_p^2 \log I_\varepsilon^\pm(v, v) \\ &= \int_{\mathbb{S}^2} \sum_{i=1}^n \sum_{j=1}^n \langle 2\pi\nabla_z G_j, \tilde{v}_j - \tilde{v}_i\rangle \langle 2\pi \nabla_z G_i, \tilde{v}_i\rangle\, \mathrm{d}\mu^\pm. \end{aligned} \tag{9}$$

Again, the formula is independent of the choice of the Killing field by the same argument as above. Moreover, if $\tilde{v}_i = \tilde{v}_*$ is the same Killing field for all i, one easily verifies that the Hessian vanishes.

Without loss of generality, we assume that the vertices of the cube, in the standard coordinates (θ, ϕ) of the sphere, are as follows:

$$\begin{aligned} p_1 &= (\theta_0, 0), & p_5 &= (\pi - \theta_0, 0),\\ p_2 &= (\theta_0, \pi/2), & p_6 &= (\pi - \theta_0, \pi/2),\\ p_3 &= (\theta_0, \pi), & p_7 &= (\pi - \theta_0, \pi),\\ p_4 &= (\theta_0, 3\pi/2), & p_8 &= (\pi - \theta_0, 3\pi/2). \end{aligned}$$

where $\theta_0 = \arccos(1/\sqrt{3})$.

We decompose the 16-dimensional space of eight tangent vectors into irreducible representtions (irreps) of the symmetry group O_h of the cube [1] as follows,

$$\bigoplus_{i=1}^{8} T_{p_i}\mathbb{S}^2 = T_{1g} \oplus T_{1u} \oplus E_g \oplus E_u \oplus T_{2g} \oplus T_{2u}.$$

Here, the irreps are:

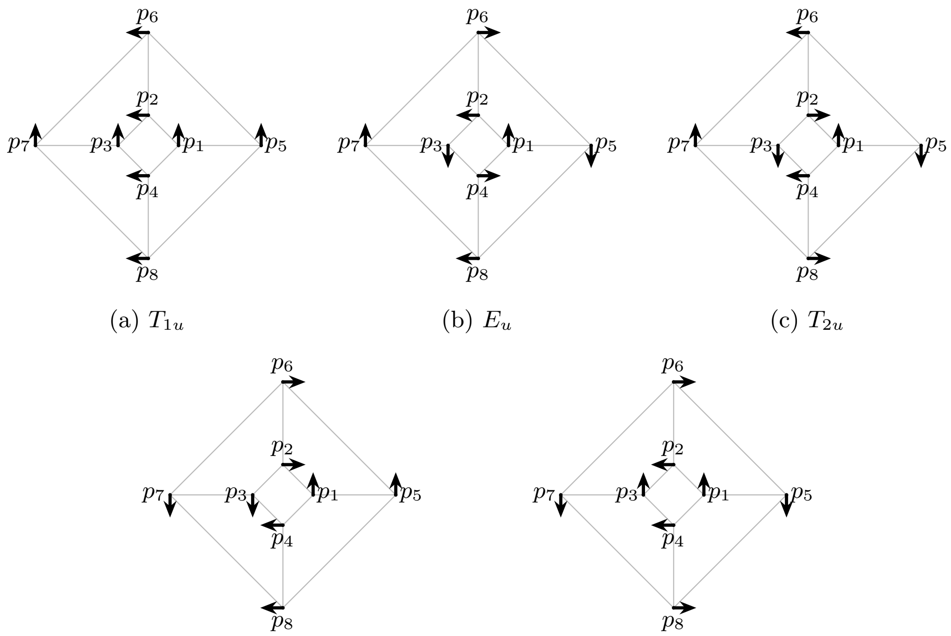
- The 3-dimensional space T_{1g} generated by global rotations. One generator is given by
$$v_i = \frac{\partial_\phi}{\sin\theta}\Big|_{p_i} \qquad 1 \le i \le 8.$$
- The 3-dimensional space T_{1u} generated by slides (that moves the center of mass). One generator is given by
$$v_i = \frac{\partial_\phi}{\sin\theta}\Big|_{p_i} \text{ for } i \in \{1,2,5,6\}, \qquad v_i = -\frac{\partial_\phi}{\sin\theta}\Big|_{p_i} \text{ for } i \in \{3,4,7,8\}.$$
- The 2-dimensional space E_g generated by the orbit of
$$v_i = \frac{\partial_\phi}{\sin\theta}\Big|_{p_i} \text{ for } i \in \{1,3,5,7\}, \qquad v_i = -\frac{\partial_\phi}{\sin\theta}\Big|_{p_i} \text{ for } i \in \{2,4,6,8\}.$$
Every face remains rectangular under these deformations.
- The 2-dimensional space E_u generated by the orbit of
$$v_i = \frac{\partial_\phi}{\sin\theta}\Big|_{p_i} \text{ for } i \in \{1,2,3,4\}, \qquad v_i = -\frac{\partial_\phi}{\sin\theta}\Big|_{p_i} \text{ for } i \in \{5,6,7,8\}.$$
The face centers of the cube remain symmetry centers under these deformations.
- The 3-dimensional space T_{2g} generated by the orbit of
$$v_i = \frac{\partial_\phi}{\sin\theta}\Big|_{p_i} \text{ for } i \in \{1,2,7,8\}, \qquad v_i = -\frac{\partial_\phi}{\sin\theta}\Big|_{p_i} \text{ for } i \in \{3,4,5,6\}.$$
- The 3-dimensional space T_{2u} generated by the orbit of
$$v_i = \frac{\partial_\phi}{\sin\theta}\Big|_{p_i} \text{ for } i \in \{1,3,6,8\}, \qquad v_i = -\frac{\partial_\phi}{\sin\theta}\Big|_{p_i} \text{ for } i \in \{2,4,5,7\}.$$

See Figure 1 for diagrams for the generators after stereographic projection.

We have seen that the Hessian is invariant under global rotations, hence T_{1g} is in the null space of the Hessian. We only need to study the restriction of the Hessian on the 13-dimensional space

$$V = \bigoplus_{i=1}^{8} T_{p_i}\mathbb{S}^2 / T_{1g} = T_{1u} \oplus E_g \oplus E_u \oplus T_{2g} \oplus T_{2u}.$$

The Hessian is an O_h-invariant symmetric bilinear form. Since the five irreducible summands are pairwise non-isomorphic and occur with multiplicity one, by Schur's lemma, the restriction of the Hessian to each summand is a scalar multiple of identity. So we only need to study the Hessian on one generator for each of the irreps. The analysis of definiteness is carried out in Appendix C. The analysis for



(a) T_{1u} (b) E_u (c) T_{2u}

(d) E_g (e) T_{2g}

FIGURE 1. Generators of the five deformation modes.

T_{2g} (Appendix C.4) is assisted by ChatGPT 5.6 Sol. The result is that the Hessian of $\log I^{\pm}$ at the cube is positive definite on all irreps except for E_u, where the Hessian is negative.

By the local rigidity theorem of Koiso–Piccione–Shoda, every sufficiently small TPMSg3 deformation of P or D, up to homotheties and Euclidean isometries, belongs to a 5-parameter family, which can only be the Meeks family. So the corresponding point configuration remains antipodal. The tangent space to the locus of antipodal configurations at the cube is $E_g \oplus T_{2g}$. Since the Hessian of $\log J_8$ is positive definite on this space, the cubic configuration is a strict local minimizer among antipodal configurations.

This finishes the proof that P and D are local minimizers of the Gauss variance. □

For the G surface, its local deformations are not fully understood. We now know a tetragonal for which the corresponding v lies in $E_u \oplus E_g$, and a rhombohedral deformation for which v lies in $T_{2u} \oplus T_{2g}$, respectively. Numerics show that G is indeed a local minimizer for the Gaussian variance along these deformations [6]. Our analysis confirms this on the rhombohedral deformation. But for the tetragonal deformation, local minimality of G means that the positive contributions of E_g outweigh the negative contribution of E_u. Indeed, numerics show that the negative eigenvalue of E_u is very close to 0.

The local deformations of G correspond to a five-dimensional subspace of the 13-dimensional V. It is very promising to hope that this subspace is sufficiently far away from E_u, then the restriction of the Hessian to the local deformations of G is positive definite, so G is indeed a local minimizer of the Gaussian variance. To confirm this, we need to work towards a better understanding of the deformations of G.

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Appendix A. The gradient

Recall that

$$\Omega_\varepsilon = \mathbb{S}^2 \setminus \cup_{i=1}^n D_\varepsilon^i,$$

where D_ε^i denote the ε-disks around p_i under the spherical metric. Define

$$I_\varepsilon^\pm = \int_{\Omega_\varepsilon} \exp(\pm u)\,\mathrm{d}A,$$

and two probability measure $\mathrm{d}\mu_\varepsilon^\pm = \rho_\varepsilon^\pm \mathrm{d}A$ on Ω_ε, where

$$\rho_\varepsilon^\pm = \frac{\exp(\pm u)}{\int_{\Omega_\varepsilon}\exp(\pm u)\,\mathrm{d}A} = \frac{\exp(\pm u)}{I_\varepsilon^\pm}.$$

We compute

$$\begin{aligned}\langle \nabla_{p_i} \log I_\varepsilon^\pm, v_i\rangle &= \frac{\langle \nabla_{p_i} I_\varepsilon^\pm, v_i\rangle}{I_\varepsilon^\pm}\\ &= \frac{1}{I_\varepsilon^\pm}\Big(\int_{\Omega_\varepsilon}\langle \pm 2\pi\nabla_{p_i} G_i, v_i\rangle \exp(\pm u)\,\mathrm{d}A + \oint_{\partial D_\varepsilon^i}\exp(\pm u)\langle n_i, \tilde{v}_i\rangle \mathrm{d}s\Big)\\ &= \int_{\Omega_\varepsilon}\langle \pm 2\pi\nabla_{p_i} G_i, v_i\rangle \rho_\varepsilon^\pm\,\mathrm{d}A + \oint_{\partial D_\varepsilon^i}\langle n_i, \tilde{v}_i\rangle \rho_\varepsilon^\pm \mathrm{d}s\end{aligned}$$

where $v_i \in T_{p_i}\mathbb{S}^2$, n_i is the unit normal vectors on the boundary of D_ε^i pointing to the interiors of the disks, and $\tilde{v}_i$ is a Killing field such that $\tilde{v}_i(p_i) = v_i$. So for

$$v = (v_1, \cdots, v_8) \in \bigoplus_{i=1}^8 T_{p_i}\mathbb{S}^2,$$

we have

$$\langle \nabla_p \log I_\varepsilon^\pm, v\rangle = \int_{\Omega_\varepsilon}\sum_{i=1}^n \langle \mp 2\pi\nabla_z G_i, \tilde{v}_i\rangle \mathrm{d}\mu_\varepsilon^\pm + \sum_{i=1}^n \oint_{\partial D_\varepsilon^i}\langle n_i, \tilde{v}_i\rangle \rho_\varepsilon^\pm \mathrm{d}s \tag{10}$$

Note that the formula does not depend on the choice of the Killing fields $\tilde{v}_i$. To see this, assume another choice $\tilde{v}_i'$. Then $\tilde{w}_i = \tilde{v}_i' - \tilde{v}_i$ is a Killing field that vanishes at p_i, which is given as a rotation around p_i. Consequently, we have $\langle \mp 2\pi\nabla_z G_i, \tilde{w}_i\rangle = 0$ and $\langle n_i, \tilde{w}_i\rangle = 0$.

Note also that, when $\tilde{v}_i = \tilde{v}_*$ is the same Killing field for all i, then

$$\begin{aligned}\langle \nabla_p \log I_\varepsilon^\pm, v\rangle &= \int_{\Omega_\varepsilon} \sum_{i=1}^n \langle \mp 2\pi \nabla_z G_i, \tilde{v}_*\rangle \mathrm{d}\mu_\varepsilon^\pm + \sum_{i=1}^n \oint_{\partial D_\varepsilon^i} \langle n_i, \tilde{v}_*\rangle \rho_\varepsilon^\pm \mathrm{d}s \\ &= \int_{\Omega_\varepsilon} \langle \mp \nabla_z u, \tilde{v}_*\rangle \mathrm{d}\mu_\varepsilon^\pm + \int_{\Omega_\varepsilon} \langle \pm \nabla_z u, \tilde{v}_*\rangle \mathrm{d}\mu_\varepsilon^\pm = 0\end{aligned}$$

So a deformation along a Killing field does not change $I_\varepsilon^\pm$.

Recall that $2\pi G_i = \log\sin(\frac{1}{2}\operatorname{dist}(z, p_i))$, so

$$2\pi\nabla_z G_i \sim \frac{\hat{w}_i}{\operatorname{dist}(z, p_i)}, \qquad \rho_\varepsilon^\pm \sim \operatorname{dist}(z, p_i)^{\pm 1}$$

as $z \to p_i$, where $\hat{w}_i = \nabla_z \operatorname{dist}(z, p_i)$. It is then obvious that (10) converges as $\varepsilon \to 0$ for μ_ε^+. As for μ_ε^-, in the standard spherical coordinate with the pole p_i, polar angle θ and azimuth angle ϕ, the integrands

$$\begin{aligned}\langle 2\pi\nabla_z G_i, \tilde{v}_i\rangle \mathrm{d}\mu_\varepsilon^- &\sim \frac{2}{\theta I^-}\langle \hat{w}_i(\theta,\phi), \tilde{v}_i(\theta,\phi)\rangle \mathrm{d}\theta\mathrm{d}\phi, \\ \langle n_i, \tilde{v}_i\rangle \rho_\varepsilon^- \mathrm{d}s &\sim \frac{-2}{I^-}\langle \hat{w}_i(\varepsilon,\phi), \tilde{v}_i(\varepsilon,\phi)\rangle \mathrm{d}\phi\end{aligned}$$

as $\theta, \varepsilon \to 0$. The integrals in (10) then converge as $\varepsilon \to 0$ because

$$\lim_{\theta\to 0}\int_0^{2\pi} \langle \hat{w}_i(\theta,\phi), \tilde{v}_i(\theta,\phi)\rangle \mathrm{d}\phi \to 0.$$

More specifically, we have

$$\langle \nabla_p \log I^\pm, v\rangle := \lim_{\varepsilon\to 0}\langle \nabla_p \log I_\varepsilon^\pm, v\rangle = \int_{\mathbb{S}^2} \sum_{i=1}^n \langle \mp 2\pi\nabla_z G_i, \tilde{v}_i\rangle \mathrm{d}\mu^\pm.$$

As a sanity check, one easily verifies that this vanishes when $\tilde{v}_i = \tilde{v}_*$ is the same Killing field for all i.

Finally, by symmetry, one easily verifies that $\langle \nabla_p \log I_\varepsilon^\pm, v\rangle = 0$ for any v when $p_1, \cdots, p_8$ are the vertices of the cube. So the cube is indeed a critical point of J_8.

Appendix B. The Hessian

We compute,

$$\begin{aligned}\langle \nabla_{p_i}\rho_\varepsilon^\pm, v_i\rangle &= \frac{\langle \pm 2\pi\nabla_{p_i} G_i, v_i\rangle \exp(\pm u)}{I_\varepsilon^\pm} - \frac{\exp(\pm u)\langle \nabla_{p_i} I_\varepsilon^\pm, v_i\rangle}{(I_\varepsilon^\pm)^2} \\ &= \rho_\varepsilon^\pm\Big(\langle \mp 2\pi\nabla_z G_i, \tilde{v}_i\rangle - \int_{\Omega_\varepsilon}\langle \mp 2\pi\nabla_z G_i, \tilde{v}_i\rangle \mathrm{d}\mu_\varepsilon^\pm - \oint_{\partial D_\varepsilon^i}\langle n_i, \tilde{v}_i\rangle \rho_\varepsilon^\pm \mathrm{d}s\Big)\end{aligned}$$

The Hessian

$$\nabla_p^2 \log I_\varepsilon^\pm(v,v) = \langle \nabla_p \langle \nabla_p \log I_\varepsilon^\pm, v\rangle, v\rangle$$

$$= \int_{\Omega_\varepsilon} \sum_{i=1}^n \langle \nabla_{p_i} \langle \pm 2\pi \nabla_{p_i} G_i, v_i\rangle, v_i\rangle \rho_\varepsilon^\pm \,\mathrm{d}A \tag{11}$$

$$+ \int_{\Omega_\varepsilon} \sum_{i=1}^n \langle \pm 2\pi \nabla_{p_i} G_i, v_i\rangle \sum_{j=1}^n \langle \nabla_{p_j} \rho_\varepsilon^\pm, v_j\rangle \,\mathrm{d}A \tag{12}$$

$$+ \sum_{i=1}^n \oint_{\partial D_\varepsilon^i} \langle n_i, \tilde{v}_i\rangle \sum_{j=1}^n \langle \nabla_{p_j} \rho_\varepsilon^\pm, v_j\rangle \mathrm{d}s \tag{13}$$

$$+ \sum_{j=1}^n \oint_{\partial D_\varepsilon^j} \langle n_j, \tilde{v}_j\rangle \sum_{i=1}^n \langle \pm 2\pi \nabla_{p_i} G_i, v_i\rangle \rho_\varepsilon^\pm \,\mathrm{d}s \tag{14}$$

$$+ \sum_{i=1}^n \oint_{\partial D_\varepsilon^i} \langle n_i, \tilde{v}_i\rangle \langle \nabla_z \rho_\varepsilon^\pm, \tilde{v}_i\rangle \mathrm{d}s$$

We expand each term

$$(11) = \int_{\Omega_\varepsilon} \sum_{i=1}^n \pm 2\pi \nabla_z^2 G_i(\tilde{v}_i, \tilde{v}_i) \mathrm{d}\mu_\varepsilon^\pm$$

$$\begin{aligned}(12) =& \int_{\Omega_\varepsilon} \sum_{i=1}^n \langle \mp 2\pi \nabla_z G_i, \tilde{v}_i\rangle \sum_{j=1}^n \langle \nabla_{p_j} \rho_\varepsilon^\pm, v_j\rangle \,\mathrm{d}A \\ =& \int_{\Omega_\varepsilon} \sum_{i=1}^n \langle \mp 2\pi \nabla_z G_i, \tilde{v}_i\rangle \sum_{j=1}^n \langle \mp 2\pi \nabla_z G_j, \tilde{v}_j\rangle \mathrm{d}\mu_\varepsilon^\pm \\ & - \int_{\Omega_\varepsilon} \sum_{i=1}^n \langle \mp 2\pi \nabla_z G_i, \tilde{v}_i\rangle \mathrm{d}\mu_\varepsilon^\pm \int_{\Omega_\varepsilon} \sum_{j=1}^n \langle \mp 2\pi \nabla_z G_j, \tilde{v}_j\rangle \mathrm{d}\mu_\varepsilon^\pm \\ & - \int_{\Omega_\varepsilon} \sum_{i=1}^n \langle \mp 2\pi \nabla_z G_i, \tilde{v}_i\rangle \mathrm{d}\mu_\varepsilon^\pm \sum_{j=1}^n \oint_{\partial D_\varepsilon^j} \langle n_j, \tilde{v}_j\rangle \rho_\varepsilon^\pm \,\mathrm{d}s\end{aligned}$$

$$\begin{aligned}(13) =& \sum_{i=1}^n \oint_{\partial D_\varepsilon^i} \langle n_i, \tilde{v}_i\rangle \sum_{j=1}^n \langle \mp 2\pi \nabla_z G_j, \tilde{v}_j\rangle \rho_\varepsilon^\pm \,\mathrm{d}s \\ & - \sum_{i=1}^n \oint_{\partial D_\varepsilon^i} \langle n_i, \tilde{v}_i\rangle \rho_\varepsilon^\pm \,\mathrm{d}s \int_{\Omega_\varepsilon} \sum_{j=1}^n \langle \mp 2\pi \nabla_z G_j, \tilde{v}_j\rangle \mathrm{d}\mu_\varepsilon^\pm \\ & - \sum_{i=1}^n \oint_{\partial D_\varepsilon^i} \langle n_i, \tilde{v}_i\rangle \rho_\varepsilon^\pm \,\mathrm{d}s \sum_{j=1}^n \oint_{\partial D_\varepsilon^j} \langle n_j, \tilde{v}_j\rangle \rho_\varepsilon^\pm \,\mathrm{d}s,\end{aligned}$$

and

$$(14) = \sum_{j=1}^n \int_{\partial D_\varepsilon^j} \langle n_j, \tilde{v}_j\rangle \sum_{i=1}^n \langle \mp 2\pi \nabla_z G_i, \tilde{v}_i\rangle \rho_\varepsilon^\pm \,\mathrm{d}s.$$

Putting them together gives

$$\nabla_p^2 \log I_\varepsilon^\pm(v,v) = \int_{\Omega_\varepsilon} \Big(\sum_{i=1}^n \langle 2\pi\nabla_z G_i, \tilde v_i\rangle\Big)^2 \mathrm{d}\mu_\varepsilon^\pm \tag{15}$$

$$+ \int_{\Omega_\varepsilon} \sum_{i=1}^n \pm 2\pi\nabla_z^2 G_i(\tilde v_i, \tilde v_i)\mathrm{d}\mu_\varepsilon^\pm \tag{16}$$

$$- \Big(\int_{\Omega_\varepsilon} \sum_{i=1}^n \langle \mp 2\pi\nabla_z G_i, \tilde v_i\rangle \mathrm{d}\mu_\varepsilon^\pm + \sum_{i=1}^n \oint_{\partial D_\varepsilon^i} \langle n_i, \tilde v_i\rangle \rho_\varepsilon^\pm \mathrm{d}s\Big)^2 \tag{17}$$

$$+ \sum_{i=1}^n \oint_{\partial D_\varepsilon^i} \langle n_i, \tilde v_i\rangle \sum_{j=1}^n \langle \mp 2\pi\nabla_z G_j, \tilde v_j\rangle \rho_\varepsilon^\pm \,\mathrm{d}s \tag{18}$$

$$+ \sum_{i=1}^n \oint_{\partial D_\varepsilon^i} \langle n_i, \tilde v_i\rangle \sum_{j=1}^n \langle \mp 2\pi\nabla_z G_j, \tilde v_j - \tilde v_i\rangle \rho_\varepsilon^\pm \,\mathrm{d}s \tag{19}$$

Note again the independence of the choice of the Killing fields $\tilde v_i$ (by the same argument as before) and, when $\tilde v_i = \tilde v_*$ is the same Killing field for all i,

$$\nabla_p^2 \log I_\varepsilon^\pm(v,v) = \int_{\Omega_\varepsilon} \langle \nabla_z u, \tilde v_*\rangle^2 \mathrm{d}\mu_\varepsilon^\pm + \int_{\Omega_\varepsilon} \pm\nabla_z^2 u(\tilde v_*, \tilde v_*)\mathrm{d}\mu_\varepsilon^\pm + \sum_{i=1}^n \oint_{\partial D_\varepsilon^i} \langle n_i, \tilde v_*\rangle\langle \mp\nabla_z u, \tilde v_*\rangle \rho_\varepsilon^\pm \,\mathrm{d}s = 0.$$

The boundary term (19) converges to 0 because D_ε^i only contains the singularity of G_j when $j = i$. For the same reason, in the limit $\varepsilon \to 0$, we may rewrite the "coupling" boundary term (18) into the form

$$\lim_{\varepsilon\to 0} \sum_{i=1}^n \oint_{\partial D_\varepsilon^i} \langle n_i, \tilde v_i\rangle \langle \mp 2\pi\nabla_z G_i, \tilde v_i\rangle \rho_\varepsilon^\pm \,\mathrm{d}s$$
$$= \lim_{\varepsilon\to 0} \sum_{i=1}^n \Big[- \int_{\Omega_\varepsilon} \langle \nabla_z u, \tilde v_i\rangle \langle 2\pi\nabla_z G_i, \tilde v_i\rangle \,\mathrm{d}\mu_\varepsilon^\pm \mp \int_{\Omega_\varepsilon} 2\pi\nabla_z^2 G_i(\tilde v_i, \tilde v_i)\,\mathrm{d}\mu_\varepsilon^\pm\Big],$$

where the second term cancels with (16).

We have seen that the second term in (17) converges to 0 as $\varepsilon \to 0$. At a critical point, its first term (17) vanishes. If this is the case, we have

$$\lim_{\varepsilon\to 0} \nabla_p^2 \log I_\varepsilon^\pm(v,v)$$
$$= \lim_{\varepsilon\to 0} \int_{\Omega_\varepsilon} \Big(\sum_{i=1}^n \langle 2\pi\nabla_z G_i, \tilde v_i\rangle\Big)^2 \mathrm{d}\mu_\varepsilon^\pm - \int_{\Omega_\varepsilon} \sum_{i=1}^n \langle \nabla_z u, \tilde v_i\rangle \langle 2\pi\nabla_z G_i, \tilde v_i\rangle \,\mathrm{d}\mu_\varepsilon^\pm$$
$$= \lim_{\varepsilon\to 0} \int_{\Omega_\varepsilon} \sum_{i=1}^n \sum_{j=1}^n \langle 2\pi\nabla_z G_j, \tilde v_j - \tilde v_i\rangle \langle 2\pi\nabla_z G_i, \tilde v_i\rangle \,\mathrm{d}\mu_\varepsilon^\pm.$$

When $i \neq j$, the convergence follows from the same argument for the convergence of the gradient (10). The possible divergence with $i = j$ is ruled out by $\tilde v_j - \tilde v_i$. In the following, we will simply write

$$\nabla_p^2 \log I^\pm(v,v) = \int_{\mathbb{S}^2} \sum_{i=1}^n \sum_{j=1}^n \langle 2\pi\nabla_z G_j, \tilde v_j - \tilde v_i\rangle \langle 2\pi\nabla_z G_i, \tilde v_i\rangle \,\mathrm{d}\mu^\pm.$$

Appendix C. Definiteness of the Hessian on the irreps

Recall from the proof the assumed positions of the vertices and the generators of each irrep.

C.1. **The definiteness on E_u and E_g.** We have chosen the generator vectors, intentionally, with the following property. The eight vertices are partitioned into two subsets A and B, with four vertices each, such that vertices in A extend to the Killing field ∂_ϕ, while vertices in B extend to $-\partial_\phi$. Then we have

$$\nabla_p^2 \log I^\pm(v,v) = \int_{\mathbb{S}^2} \sum_{i=1}^n \sum_{j=1}^n \langle 2\pi\nabla_z G_j, \tilde{v}_j - \tilde{v}_i\rangle \langle 2\pi\nabla_z G_i, \tilde{v}_i\rangle \, \mathrm{d}\mu^\pm$$
$$= -4\int_{\mathbb{S}^2} \sum_{i\in A}\sum_{j\in B} (2\pi\partial_\phi G_j)(2\pi\partial_\phi G_i)\, \mathrm{d}\mu^\pm = -16\pi^2 \int_{\mathbb{S}^2} \partial_\phi G_A \partial_\phi G_B \, \mathrm{d}\mu^\pm,$$

where

$$G_A = \sum_{i\in A} G_i, \qquad G_B = \sum_{i\in B} G_i.$$

One then easily observes that $\partial_\phi G_A \partial_\phi G_B$ is everywhere non-positive for E_g, and everywhere non-negative for E_u. Since the integrand is not identically zero, this proves that the Hessian is positive definite on E_g, and negative definite on E_u.

C.2. **The definiteness on T_{1u}.** To study the definiteness of on T_{1u}, we need to first study the sign of

$$H_{ij}^\pm = \int_{\mathbb{S}^2} \partial_\phi G_i \partial_\phi G_j \, \mathrm{d}\mu^\pm.$$

When p_i and p_j are antipodal (e.g. $(i,j) = (1,7)$) or on a horizontal face diagonal (e.g. $(i,j) = (1,3)$), one easily sees that the integrand $\partial_\phi G_i \partial_\phi G_j \le 0$ everywhere on the sphere, so $H_{ij}^\pm$ is negative. When p_i and p_j are vertical neighbors (e.g. $(i,j) = (1,5)$), one easily sees that the integrand $\partial_\phi G_i \partial_\phi G_j \ge 0$ everywhere on the sphere, so $H_{ij}^\pm$ is positive. Otherwise, the following lemma would be useful:

Lemma. *Let $f(\theta,\phi)$ be a function defined on $\mathbb{S}^2$ that is odd in ϕ, and ρ be a positive function on $\mathbb{S}^2$ with*

$$\rho(\theta,\phi) = \rho(\theta, \phi + \tfrac{\pi}{2}) = \rho(\pi - \theta, \phi).$$

Assume that, for every $0 < \theta < \pi$ and $0 < \phi < \pi/2$, we have

$$f(\theta,\phi) > f(\theta, \pi - \phi),$$

Then the integrals

$$K_1 = \int_{\mathbb{S}^2} f(\theta,\phi) f(\theta, \phi - \tfrac{\pi}{2}) \rho \, \mathrm{d}A < 0,$$
$$K_2 = \int_{\mathbb{S}^2} f(\theta,\phi) f(\pi - \theta, \phi - \tfrac{\pi}{2}) \rho \, \mathrm{d}A < 0.$$

Proof. We decompose the sphere into four quadrants. Then

$$K_1 = \int_0^\pi \sin\theta \, d\theta \int_0^{\frac{\pi}{2}} \rho \, d\phi \big(f(\theta,\phi) f(\theta,\phi - \tfrac{\pi}{2}) + f(\theta, \phi + \tfrac{\pi}{2}) f(\theta,\phi)$$
$$+ f(\theta, \phi + \pi) f(\theta, \phi + \tfrac{\pi}{2}) + f(\theta, \phi - \tfrac{\pi}{2}) f(\theta, \phi + \pi)\big)$$
$$= \int_0^\pi \sin\theta \, d\theta \int_0^{\frac{\pi}{2}} \rho \, d\phi \big(f(\theta,\phi) + f(\theta,\phi+\pi)\big)\big(f(\theta, \phi - \tfrac{\pi}{2}) + f(\theta, \phi + \tfrac{\pi}{2})\big)$$
$$= \int_0^\pi \sin\theta \, d\theta \int_0^{\frac{\pi}{2}} \rho \, d\phi \big(f(\theta,\phi) - f(\theta,\pi - \phi)\big)\big(- f(\theta, \tfrac{\pi}{2} - \phi) + f(\theta, \tfrac{\pi}{2} + \phi)\big)$$

For $0 < \phi < \frac{\pi}{2}$, by the assumed inequalities, K_1 is negative. On the other hand,

$$\begin{aligned}
K_2 = \int_0^{\frac{\pi}{2}} \sin\theta\, d\theta \int_0^{\frac{\pi}{2}} \rho\, d\phi \big(& f(\theta,\phi) f(\pi-\theta,\phi-\tfrac{\pi}{2}) + f(\pi-\theta,\phi) f(\theta,\phi-\tfrac{\pi}{2}) \\
& + f(\theta,\phi+\tfrac{\pi}{2}) f(\pi-\theta,\phi) + f(\pi-\theta,\phi+\tfrac{\pi}{2}) f(\theta,\phi) \\
& + f(\theta,\phi+\pi) f(\pi-\theta,\phi+\tfrac{\pi}{2}) + f(\pi-\theta,\phi+\pi) f(\theta,\phi+\tfrac{\pi}{2}) \\
& + f(\theta,\phi-\tfrac{\pi}{2}) f(\pi-\theta,\phi+\pi) + f(\pi-\theta,\phi-\tfrac{\pi}{2}) f(\theta,\phi+\pi)\big) \\
= \int_0^{\frac{\pi}{2}} \sin\theta\, d\theta \int_0^{\frac{\pi}{2}} \rho\, d\phi & \\
& \Big[\big(f(\pi-\theta,\phi) + f(\pi-\theta,\phi+\pi)\big)\big(f(\theta,\phi-\tfrac{\pi}{2}) + f(\theta,\phi+\tfrac{\pi}{2})\big) \\
& + \big(f(\theta,\phi) + f(\theta,\phi+\pi)\big)\big(f(\pi-\theta,\phi-\tfrac{\pi}{2}) + f(\pi-\theta,\phi+\tfrac{\pi}{2})\big)\Big] \\
= \int_0^{\frac{\pi}{2}} \sin\theta\, d\theta \int_0^{\frac{\pi}{2}} \rho\, d\phi & \\
& \Big[\big(f(\pi-\theta,\phi) - f(\pi-\theta,\pi-\phi)\big)\big(-f(\theta,\tfrac{\pi}{2}-\phi) + f(\theta,\phi+\tfrac{\pi}{2})\big) \\
& + \big(f(\theta,\phi) - f(\theta,\pi-\phi)\big)\big(-f(\pi-\theta,\tfrac{\pi}{2}-\phi) + f(\pi-\theta,\phi+\tfrac{\pi}{2})\big)\Big]
\end{aligned}$$

and the integrand is negative by the assumed inequalities. □

We now verify that the assumption on f is satisfied by $\partial_\phi G_1$. In fact,

$$G_1(\theta,\phi) = \frac{1}{4\pi}\log(1 - \cos\theta\cos\theta_0 - \sin\theta\sin\theta_0\cos\phi),$$

so

$$\partial_\phi G_1 = \frac{b\sin\phi}{4\pi(a - b\cos\phi)},$$

where $a = 1 - \cos\theta\cos\theta_0$, $b = \sin\theta\sin\theta_0$. Then we compute

$$\begin{aligned}
\partial_\phi G_1(\theta,\phi) - \partial_\phi G_1(\theta,\pi-\phi) =& \frac{b\sin\phi}{4\pi}\Big(\frac{1}{a - b\cos\phi} - \frac{1}{a + b\cos\phi}\Big) \\
=& \frac{b^2\sin\phi\cos\phi}{2\pi(a^2 - b^2\cos^2\phi)}.
\end{aligned}$$

This is positive because

$$\begin{aligned}
(a^2 - b^2\cos^2\phi) = a^2 - b^2 + b^2\sin^2\phi &= (a-b)(a+b) + b^2\sin^2\phi \\
&= (1 - \cos(\theta-\theta_0))(1 - \cos(\theta+\theta_0)) + b^2\sin^2\phi > 0.
\end{aligned}$$

So we can apply the lemma on $\rho = \rho^\pm$ and $f = \partial_\phi G_1$. We conclude by symmetry that, whenever p_i and p_j are horizontal neighbors (e.g. $(i,j) = (1,2)$) or on a vertical face diagonal (e.g. $(i,j) = (1,6)$), we have $H_{ij}^\pm < 0$.

For T_{1u}, the integrand does not involve any pair of vertical neighbors, so we can already conclude that the Hessian is positive definite on the irreps T_{1u}.

C.3. **The definiteness on T_{2u}.** For T_{2u} we place the cube so that a pair of antipodal vertices are at the north and the south pole. So

$$\begin{aligned}
p_1 &= (0,0), & p_5 &= (\pi,0), \\
p_2 &= (\theta_0,0), & p_6 &= (\pi-\theta_0,\pi/3), \\
p_3 &= (\theta_0,2\pi/3), & p_7 &= (\pi-\theta_0,\pi), \\
p_4 &= (\theta_0,4\pi/3), & p_8 &= (\pi-\theta_0,5\pi/3).
\end{aligned}$$

where $\theta_0 = \arccos(-1/3)$. Then a generator for T_{2u} is given by

$$v_i = \frac{\partial_\phi}{\sin\theta}\Big|_{p_i} \text{ for } i \in \{2,3,4\}, \qquad v_i = -\frac{\partial_\phi}{\sin\theta}\Big|_{p_i} \text{ for } i \in \{6,7,8\}, \qquad v_1 = v_5 = 0.$$

Note that the vertices p_1 and p_5 (at the poles) are actually fixed under this generator. The Hessian becomes

$$\nabla_p^2 \log I^\pm(v,v) = -16\pi^2 \int_{\mathbb{S}^2} \partial_\phi G_A \partial_\phi G_B \,\mathrm{d}\mu^\pm,$$

where $A = \{2,3,4\}$ and $B = \{6,7,8\}$. One easily notices that the integrand $\partial_\phi G_A \partial_\phi G_B \le 0$ everywhere on the sphere. So the Hessian is positive definite on T_{2u}.

C.4. **The definiteness on T_{2g}.** This part contains the most involved computation. The argument was found with the assistance of ChatGPT 5.6 Sol.

C.4.1. *Preparation.* Given a point $p \in \mathbb{S}^2$ with polar angle θ_0 and azimuth angle ϕ_0, the Green function $G(\theta,\phi)$ centered at p can be written as (compare (6))

$$\frac{1}{4\pi} \log\big(1 - \cos\theta\cos\theta_0 - \sin\theta\sin\theta_0\cos(\phi-\phi_0)\big).$$

After grouping the eight points into four antipodal pairs, we obtain

$$\begin{aligned}
\widehat{G}_1 &= G_1 + G_7 = \frac{1}{4\pi}\log\frac{3-\zeta_1^2}{3}, & \zeta_1 &= \sqrt{2}x + z,\\
\widehat{G}_2 &= G_2 + G_8 = \frac{1}{4\pi}\log\frac{3-\zeta_2^2}{3}, & \zeta_2 &= \sqrt{2}y + z,\\
\widehat{G}_3 &= G_3 + G_5 = \frac{1}{4\pi}\log\frac{3-\zeta_3^2}{3}, & \zeta_3 &= \sqrt{2}x - z,\\
\widehat{G}_4 &= G_4 + G_6 = \frac{1}{4\pi}\log\frac{3-\zeta_4^2}{3}, & \zeta_4 &= \sqrt{2}y - z,
\end{aligned}$$

where

$$x = \sin\theta\cos\phi, \qquad y = \sin\theta\sin\phi, \qquad z = \cos\theta.$$

Then we have

$$\exp(u) = \exp\Big(2\pi\sum \widehat{G}_i\Big) = \frac{\sqrt{(3-\zeta_1^2)(3-\zeta_2^2)(3-\zeta_3^2)(3-\zeta_4^2)}}{9},$$

and

$$\begin{aligned}
\frac{\partial_\phi}{\sin\theta}\widehat{G}_1 &= \frac{\zeta_1\sin\phi}{\sqrt{2}\pi(3-\zeta_1^2)}, & \frac{\partial_\phi}{\sin\theta}\widehat{G}_2 &= \frac{-\zeta_2\cos\phi}{\sqrt{2}\pi(3-\zeta_2^2)},\\
\frac{\partial_\phi}{\sin\theta}\widehat{G}_3 &= \frac{\zeta_3\sin\phi}{\sqrt{2}\pi(3-\zeta_3^2)}, & \frac{\partial_\phi}{\sin\theta}\widehat{G}_4 &= \frac{-\zeta_4\cos\phi}{\sqrt{2}\pi(3-\zeta_4^2)}.
\end{aligned}$$

So we need to prove that

$$H^+ = \int_0^\pi \mathrm{d}\theta \int_0^{2\pi} \frac{\mathrm{d}\phi}{18\pi^2}\frac{P_1}{\sqrt{1-z^2}Q^{1/2}}, \qquad H^- = \int_0^\pi \mathrm{d}\theta \int_0^{2\pi} \frac{9\mathrm{d}\phi}{2\pi^2}\frac{P_1}{\sqrt{1-z^2}Q^{3/2}}$$

are both negative, where

$$\begin{aligned}
P_1 &= (\zeta_1 y(3-\zeta_2^2) - \zeta_2 x(3-\zeta_1^2))(\zeta_3 y(3-\zeta_4^2) - \zeta_4 x(3-\zeta_3^2)),\\
Q &= (3-\zeta_1^2)(3-\zeta_2^2)(3-\zeta_3^2)(3-\zeta_4^2).
\end{aligned}$$

C.4.2. *Symmetrization.* The integrals are invariant under the symmetries

- $x \leftrightarrow -x$ that induces $\zeta_1 \leftrightarrow -\zeta_3$,
- $y \leftrightarrow -y$ that induces $\zeta_2 \leftrightarrow -\zeta_4$,
- $z \leftrightarrow -z$ that induces $(\zeta_1, \zeta_2) \leftrightarrow (\zeta_3, \zeta_4)$, and
- $x \leftrightarrow y$ that induces $(\zeta_1, \zeta_3) \leftrightarrow (\zeta_2, \zeta_4)$.

After symmetrization, we have

$$H^+ = 8 \int_0^{\pi/2} \mathrm{d}\theta \int_0^{\pi/4} \frac{\mathrm{d}\phi}{18\pi^2} \frac{P}{\sqrt{1-z^2}Q^{1/2}},$$
$$H^- = 8 \int_0^{\pi/2} \mathrm{d}\theta \int_0^{\pi/4} \frac{9\mathrm{d}\phi}{2\pi^2} \frac{P}{\sqrt{1-z^2}Q^{3/2}},$$

where $P = P_1 + P_2$ and

$$P_2 = (\zeta_3 y(3-\zeta_2^2) - \zeta_2 x(3-\zeta_3^2))(\zeta_1 y(3-\zeta_4^2) - \zeta_4 x(3-\zeta_1^2)).$$

Define $s = x^2 + y^2 = 1 - z^2$ and $t = x^2 - y^2 = s\cos(2\phi)$. Then P and Q can be written as polynomials of s and t as follows.

$$P = -4t^4 + 2(7s^2 - 25s + 20)t^2 + 8(s^2+s)(s^2-1)$$
$$Q = ((2-t)^2 - 4(s+t)(1-s))((2+t)^2 - 4(s-t)(1-s)),$$

This is first suggested by ChatGPT 5.6 Sol, then verified by Mathematica and manually by the author.

C.4.3. *Bounding P and Q.* We point out the following bounds of P and Q:

$P > 0$ only when $s > \frac{8}{9}$: Write

$$P = -4\tau^2 + 2\tau(7s^2 - 25s + 20) + 8s(s+1)(s^2-1)$$

with $\tau = t^2$. We compute the partial derivative

$$\frac{\partial P}{\partial \tau} = -8\tau + 2(7s^2 - 25s + 20).$$

For $0 \le t \le s \le 8/9$, we have

$$\frac{\partial P}{\partial \tau} \ge 2(3s^2 - 25s + 20) > 0,$$

so

$$P(t,s) \le P(s,s) = 2s(2-3s)^2(s-1) \le 0.$$

This proves that $P \le 0$ when $s \le 8/9$.

$P \le s$ when $\frac{8}{9} \le s \le 1$:

$$\frac{P}{s} = -4\frac{\tau^2}{s} + 2\frac{\tau}{s}(7s^2 - 25s + 20) + 8(s+1)(s^2-1)$$

has an unrestricted maximum

$$8(s+1)(s^2-1) + \frac{(7s^2 - 25s + 20)^2}{4s},$$

which is indeed ≤ 1 for $8/9 \le s \le 1$.

$Q \le 16$: Write

$$Q = \tau^2 - 8\tau(s^2 - 7s + 7) + 16(s^2 - s + 1)^2$$

with $\tau = t^2$. Then the partial derivative

$$\frac{\partial Q}{\partial \tau} = 2\tau - 8(s^2 - 7s + 7) \tag{20}$$

is negative for $0 \le t \le s \le 1$. So we have

$$Q(t,s) \le Q(0,s) = 16(s^2 - s + 1)^2 \le 16,$$

with equality when $t = 0$ and $s = 0$ or 1.

$Q \geq (\frac{52}{27})^2$ when $s \geq \frac{8}{9}$: As the partial derivative (20) is negative, we have for $0 \leq t \leq s$ and $8/9 \leq s \leq 1$, we have

$$Q(t,s) \geq Q(s,s) = (-4+4s+3s^2)^2 \geq \Big(\frac{52}{27}\Big)^2,$$

with equality when $t = s = 8/9$.

C.4.4. *Bounding the integrals.* Finally, we estimate the integrals

$$H^+ = 8\int_0^1 \mathrm{d}z \int_0^{\pi/4} \frac{\mathrm{d}\phi}{18\pi^2}\frac{P}{sQ^{1/2}}, \qquad H^- = 8\int_0^1 \mathrm{d}z \int_0^{\pi/4} \frac{9\mathrm{d}\phi}{2\pi^2}\frac{P}{sQ^{3/2}}.$$

Write $P^+ = \max(P,0)$. Then

$$\frac{P}{\sqrt{Q}^\alpha} \leq \frac{P}{4^\alpha} + \Big((\frac{27}{52})^\alpha - \frac{1}{4^\alpha}\Big)P^+.$$

where $\alpha = 1$ for H^+ and $\alpha = 3$ for H^-. Therefore,

$$\int_0^1 \mathrm{d}z \int_0^{\pi/4} \mathrm{d}\phi \frac{P}{s\sqrt{Q}^\alpha} \leq \frac{1}{4^\alpha}\int_0^1 \mathrm{d}z \int_0^{\pi/4} \mathrm{d}\phi \frac{P}{s} \tag{21}$$

$$+ \Big((\frac{27}{52})^\alpha - \frac{1}{4^\alpha}\Big)\int_0^1 \mathrm{d}z \int_0^{\pi/4} \mathrm{d}\phi \frac{P^+}{s} \tag{22}$$

The integral on the right-hand-side of (21) can be explicitly computed

$$\int_0^1 \mathrm{d}z \int_0^{\pi/4} \mathrm{d}\phi \frac{P}{s} = -\frac{76}{105}\pi.$$

The integral (22) can be restricted to the region $8/9 \leq s \leq 1$, where we know that $P/s \leq 1$, so

$$\int_0^1 \mathrm{d}z \int_0^{\pi/4} \mathrm{d}\phi \frac{P^+}{s} = \int_0^{1/3} \mathrm{d}z \int_0^{\pi/4} \mathrm{d}\phi \frac{P^+}{s} \leq \frac{\pi}{12}.$$

Putting these all together yields

$$\int_0^1 \mathrm{d}z \int_0^{\pi/4} \mathrm{d}\phi \frac{P}{s\sqrt{Q}^\alpha} \leq -\frac{1}{4^\alpha}\frac{76}{105}\pi + \Big((\frac{27}{52})^\alpha - \frac{1}{4^\alpha}\Big)\frac{\pi}{12}.$$

This is negative for $\alpha = 1$ and $\alpha = 3$, thereby proving that $H^\pm$ are both negative.

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